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Continuity and Strong Unicity of the Best Approximation Operator on Subintervals

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1. INTRODUCTION

Let π_n denote the set of real algebraic polynomials of degree n or less. For a given closed interval I of the real line let $C(I)$ denote the set of continuous real-valued functions on I endowed with the uniform norm $\|\cdot\|_I$. For a fixed positive integer n, we define the best uniform approximation $P_{f,I}$ to $f \in C(I)$ from π_n by $\|f-P_{f,I}\|_I = \inf\{\|f-P\|_I \mid P \in \pi_n\}$ and the degree of approximation $E_n(f; a, b)$ to f from π_n on the interval $I=[a,b]$ by $E_n(f; a, b) = \|f-P_{f,[a,b]}\|_{[a,b]}$. We assume that n is fixed (here and throughout the paper) and that approximation is from π_n, except in Theorem 2.3, where we replace π_n by a finite dimensional Haar subspace H. The definition of best approximation from H is analogous. The standard results concerning strong unicity and Lipschitz constants can be found in Cheney [2, pp. 80–82] and are stated in the following theorem.

THEOREM 1.1. *Let $f \in C(I)$. Then there are constants $\lambda_{f,I} > 0$ and $\gamma_{f,I} > 0$ such that*

$$\|P_{f,I} - P_{g,I}\|_I \leqslant \lambda_{f,I} \|f-g\|_I \tag{1.1}$$

for all $g \in C(I)$, and

$$\|f-P_{f,I}\|_I \leqslant \|f-Q\|_I - \gamma_{f,I} \|Q-P_{f,I}\|_I \tag{1.2}$$

for all $Q \in \pi_n$.

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We note that $\lambda_{f,I}$ is called a uniform Lipschitz constant and $\gamma_{f,I}$ is called a strong unicity constant. Expression (1.2) is called the strong unicity inequality. We further note that if $\gamma_{f,I} > 0$ is known, then an acceptable value of $\lambda_{f,I}$ is $2/\gamma_{f,I}$. See Cheney [2, p. 82].

In [4], Henry and Roulier investigate the existence of uniform Lipschitz constants on all symmetric intervals of the form $[-\alpha, \alpha] \subset [-1, 1]$ for a given $f \in C[-1, 1]$. Sufficient conditions on f are obtained to guarantee the existence of a constant $\lambda_f > 0$ so that

$$\|P_{f,J} - P_{g,J}\|_J \leqslant \lambda_f \|f - g\|_J \tag{1.3}$$

for all $g \in C(J)$ and for all $J \subset [-1, 1]$ of the form $J = [-\alpha, \alpha]$. Examples are also given of functions $f \in C(I)$ which fail to have such λ_f.

In this paper we present sufficient conditions on $f \in C(I)$ to ensure the existence of a strong unicity constant $\lambda_f > 0$ valid for all closed subintervals of I. This, in turn, guarantees that (1.3) is valid for all closed subintervals J of I.

2. The Main Theorems

The proofs of Theorems 2.2 and 2.5 employ techniques similar to those used in the proof of Theorem 3.1 [4, p. 228] and all of these theorems make use of the following lemma due to Cline [3]. (See also [1].)

Lemma 2.1. *Let $h \in C(I)$ with $h \notin \pi_n$. Let $P \in \pi_n$ be the best approximation to h on I and for each Chebyshev alternation $E = \{t_j\}_{j=1}^{n+2}$ for $h - P$, define $q_i \in \pi_n$ by $q_i(t_j) = \operatorname{sgn}[h(t_j) - P(t_j)]$, $j = 1, 2, \ldots, n+2$, $j \neq i$ and $i = 1, 2, \ldots, n+2$. Let $\Omega(E) = \max_{1 \leqslant i \leqslant n+2}\{\|q_i\|_I\}$. Then there exists a Chebyshev alternation E^* for $h - P$ so that*

$$\lambda_{h,I} \leqslant 2\Omega(E^*), \tag{2.1}$$

where $\lambda_{h,I}$ is the Lipschitz constant for h on I and so that

$$[\gamma_{h,I}]^{-1} \leqslant \Omega(E^*), \tag{2.2}$$

where $\gamma_{h,I}$ is the strong unicity constant for h on I.

Theorem 2.2. *If $f \in C^{n+1}[-1, 1]$ with $f^{(n+1)}(x) > 0$ on $[-1, 1]$, then there are positive constants λ_f and γ_f so that for all closed subintervals $J \subset [-1, 1]$,*

$$\|P_{f,J} - P_{g,J}\|_J \leqslant \lambda_f \|f - g\|_J \tag{2.3}$$

for all $g \in C(J)$, *and*

$$\|f - P_{f,J}\|_J \leqslant \|f - Q\|_J - \gamma_f \|Q - P_{f,J}\|_J \tag{2.4}$$

for all $Q \in \pi_n$.

Proof. If $f^{(n+1)}(x) > 0$ on $[-1, 1]$ then $f \notin \pi_n$ for any subinterval $J \subset [-1, 1]$. Thus for a given $k \geqslant n$ there exist $p \in \pi_k$ and positive numbers m and M so that

$$mp^{(n+1)}(x) \leqslant f^{(n+1)}(x) \leqslant Mp^{(n+1)}(x) \tag{2.5}$$

for all $x \in [-1, 1]$.

By Bernstein's Theorem [7, p. 38],

$$mE_n(p; a, b) = E_n(mp; a, b) \leqslant E_n(f; a, b) \tag{2.6}$$

for any $[a, b] \subset [-1, 1]$. Let

$$e(p; a, b) = p(x) - P_{p,[a,b]}(x).$$

Then

$$\|e(p; a, b)\|_{[a,b]} = E_n(p; a, b)$$

and

$$e^{(n+1)}(p; a, b)(x) = p^{(n+1)}(x).$$

Now Markoff's inequality [2, pp. 91, 94] implies

$$|e^{(n+1)}(p; a, b)(x)| \leqslant \frac{2^{n+1}k^{2n+2}}{(b-a)^{n+1}} E_n(p; a, b)$$

for all $x \in [a, b]$. Thus

$$\|p^{(n+1)}\|_{[a,b]} \leqslant \frac{2^{n+1}k^{2n+2}}{(b-a)^{n+1}} E_n(p; a, b), \tag{2.7}$$

Let $E_{[a,b]} = \{t_j\}_{j=1}^{n+2}$ be any Chebyshev alternation for

$$d(a, b, f)(x) = [f - P_{f,[a,b]}](x).$$

If $\{q_i\}_{i=1}^{n+2}$ is the set of polynomials of Lemma 2.1 for the Chebyshev alternation $E_{[a,b]}$ then

$$q_i(t_j) = \frac{d(a, b, f)(t_j)}{E_n(f; a, b)}, \qquad j = 1, 2, \ldots, n+2, j \neq i, \; i = 1, 2, \ldots, n+2.$$

Then the classical remainder theorem of interpolation theory [2, p. 60] implies

$$\frac{d(a,b,f)(x)}{E_n(f;a,b)} - q_i(x) = \frac{d^{(n+1)}(a,b,f)(\xi)\, w_i(x)}{E_n(f;a,b)(n+1)!},$$

where x, $\xi \in [a,b]$ and

$$w_i(x) = \prod_{\substack{j=1\\ j\neq i}}^{n+2} (x - t_j).$$

But $d^{(n+1)}(a,b,f)(\xi) = f^{(n+1)}(\xi)$. Thus, from this and (2.5) we have

$$\left|\frac{d(a,b,f)(x)}{E_n(f;a,b)} - q_i(x)\right| \leqslant \frac{Mp^{(n+1)}(\xi)\,|w_i(x)|}{E_n(f;a,b)(n+1)!}.$$

So from (2.6) and (2.7) we have

$$\begin{aligned} |q_i(x)| &\leqslant \frac{Mp^{(n+1)}(\xi)\,|w_i(x)|}{E_n(f;a,b)(n+1)!} + 1 \\ &\leqslant \frac{Mp^{(n+1)}(\xi)(b-a)^{n+1}}{mE_n(p;a,b)(n+1)!} + 1 \\ &\leqslant \frac{M2^{n+1}k^{2n+2}}{m(n+1)!} + 1. \end{aligned}$$

Thus

$$\max_{1\leqslant i\leqslant n+2} |q_i(x)| \leqslant \frac{M2^{n+1}k^{2n+2}}{m(n+1)!} + 1$$

and so

$$\lambda_{f,J} \leqslant \frac{M2^{n+2}k^{2n+2}}{m(n+1)!} + 2$$

and

$$\gamma_{f,J} \geqslant \left[\frac{M2^{n+1}k^{2n+2}}{m(n+1)!} + 1\right]^{-1}$$

for any $J \subset [-1, 1]$. Our conclusions then follow.

The strong Kolmogorov criterion [1, p. 246] states for H a finite dimensional Haar subspace of $C(I)$ and $f \in C(I) \backslash H$ that

$$\gamma_{f,I} = \inf_{h \in S(H)} \max_{x \in E(f)} [f(x) - P_{f,I}(x)] \, \|f - P_{f,I}\|^{-1} \, h(x),$$

where $S(H) = \{h \in H \mid \|h\| = 1\}$ and

$$E(f) = \{x \in I \mid |f(x) - P_{f,I}(x)| = \|f - P_{f,I}\|_I\}.$$

$P_{f,I}$ is defined as in Section 1 with π_n replaced by H.

We assume, for Theorem 2.3, that approximation is from a finite dimensional Haar subspace (see [4, p. 224]). The first part of Theorem 2.3 (2.8) is due to Henry and Schmidt [5] and we have proved (2.9) which is a similar result for strong unicity constants.

THEOREM 2.3. *If Γ is a compact subset of $C(I)$ and $\Gamma \cap H = \phi$, then there are constants $\lambda_\Gamma > 0$ and $\gamma_\Gamma > 0$ so that*

$$\|P_{f,I} - P_{g,I}\|_I \leqslant \lambda_\Gamma \, \|f - g\|_I \tag{2.8}$$

for all $f \in \Gamma$ and $g \in C(I)$, and

$$\|f - P_{f,I}\|_I \leqslant \|f - Q\|_I - \gamma_\Gamma \, \|Q - P_{f,I}\|_I \tag{2.9}$$

for $f \in \Gamma$ and $Q \in H$.

Proof of (2.9). Suppose no such γ_Γ exists. Then there is a sequence $\{f_n\}$ so that $f_n \in \Gamma$ and $\lim_{n \to \infty} \gamma_{f_n} = 0$. Let $x_{0,n} < x_{1,n} < \cdots < x_{k,n}$ be a Chebyshev alternation for f_n. Then the strong Kolmogorov criterion implies

$$\limsup_{n \to \infty} [\inf_{h \in S(H)} (\max_{0 \leqslant j \leqslant n} (\sigma(f_n, x_{j,n}) \, h(x_{j,n})))] = 0,$$

where $\sigma(g, x) = [g(x) - P_{g,I}(x)] \, \|g - P_{g,I}\|_I^{-1}$. Thus there is a sequence $\{h_n\}_{n=0}^{\infty}$, $h_n \in S(H)$ so that for $j = 0, 1,..., n$

$$\limsup_{n \to \infty} [\max_{0 \leqslant j \leqslant k} \sigma(f_n, x_{j,n}) \, h_n(x_{j,n})] \leqslant 0.$$

Γ and $S(H)$ are compact so we can assume, without loss of generality that $\lim_{n \to \infty} f_n = f \in \Gamma$ and $\lim_{n \to \infty} h_n = S(H)$. Then for $j = 0, 1,..., k$, $\limsup_{n \to \infty} \max_{0 \leqslant j < k} \sigma(f_n, x_{j,n}) \, h(x_{j,n}) \leqslant 0$. Furthermore, $\sigma(f_n, \cdot) \to \sigma(f, \cdot)$ uniformly on I. But, we may also assume $\lim_{n \to \infty} x_{j,n} = x_j$ for $j = 0, 1,..., k$ and $x_0 < x_1 < \cdots < x_k$. (Otherwise, if $x_j = x_{j+1}$ for some j, then $\lim_{n \to \infty} x_{j,n} = \lim_{n \to \infty} x_{j+1,n} = x_j$ and $\lim_{n \to \infty} \sigma(f_n, x_{j,n}) = \sigma(f, x_j)$, $\lim_{n \to \infty} \sigma(f_n, x_{j+1,n}) =$

$\sigma(f, x_j)$ and $\lim_{n\to\infty}[\sigma(f_n, x_{j,n}) \cdot \sigma(f_n, x_{j+1,n})] = -1 = \sigma^2(f, x_j)$ which is impossible.) Thus

$$\limsup_{n\to\infty} \max_{0\leqslant j\leqslant k} \sigma(f_n, x_j)\, h(x_j) \leqslant 0.$$

Since $\sigma(f_n, \cdot) \to \sigma(f, \cdot)$ uniformly on I we have

$$\max_{0\leqslant j\leqslant k} (\sigma(f, x_j)\, h(x_j)) \leqslant 0.$$

Now $\sigma(f, x_j) \cdot \sigma(f, x_{j+1}) = -1$, $j = 0, 1,..., k-1$ so $h(x_j)$ also changes sign k times. But then $h = 0$ which is a contradiction.

LEMMA 2.4. *Let $f \in C[-1, 1]$. Suppose $\varepsilon > 0$ and there does not exist a closed interval $I \subset [-1, 1]$ so that the length of I, $l(I) \geqslant \varepsilon$ and f restricted to I is in π_n. Then there are constants $\lambda_f(\varepsilon) > 0$ and $\gamma_f(\varepsilon) > 0$ so that for every closed interval $J \subset [-1, 1]$ which satisfies $l(J) \geqslant \varepsilon$,*

$$\|P_{f,J} - P_{g,J}\|_J \leqslant \lambda_f(\varepsilon)\, \|f - g\|_J \tag{2.10}$$

for all $g \in C(J)$ and

$$\|f - P_{f,J}\|_J \leqslant \|f - Q\|_J - \gamma_f(\varepsilon)\, \|Q - P_{f,J}\|_J \tag{2.11}$$

for all $Q \in \pi_n$.

Proof. Suppose such a $\lambda_f(\varepsilon)$ as in (2.10) does not exist. Then for each positive integer k, there is a closed interval, $J_k \subset [-1, 1]$, $l(J_k) \geqslant \varepsilon$ and $g_k \in C(J_k)$ so that

$$\|P_{f,J_k} - P_{g_k,J_k}\|_{J_k} > k\, \|f - g_k\|_{J_k}. \tag{2.12}$$

If a $\gamma_f(\varepsilon)$ as in (2.11) does not exist, then for each positive integer k there is a closed interval $J_k \subset [-1, 1]$, $l(J_k) \geqslant \varepsilon$ and $p_k \in \pi_n$ so that

$$\|f - P_{f,J_k}\|_{J_k} > \|f - p_k\|_{J_k} - 1/k\, \|p_k - P_{f,J_k}\|_{J_k}. \tag{2.13}$$

Denote $J_k = [a_k, b_k]$ and define

$$f_k(x) = f\left(a_k + \frac{x+1}{2}(b_k - a_k)\right) \in C[-1, 1],$$

$$\hat{g}_k(x) = g_k\left(a_k + \frac{x+1}{2}(b_k - a_k)\right) \in C[-1, 1],$$

$$\hat{p}_k(x) = p_k\left(a_k + \frac{x+1}{2}(b_k - a_k)\right) \in C[-1, 1]$$

for every positive integer k. We can choose a subsequence of the J_k's, call it $\{J_{k_j}\}$, such that $[a_{k_j}, b_{k_j}] \to [a, b] \subset [-1, 1]$. Then $l([a, b]) \geqslant \varepsilon$ and $\{f_{k_j}\}$ will converge to $\hat{f} \in C[-1, 1]$, where $\hat{f}(x) = f(a + ((x+1)/2)(b-a))$. Let $\Gamma = \{f_{k_j}\} \cup \{\hat{f}\}$. To prove (2.10) we note that Γ is sequentially compact and $\Gamma \cap \pi_n = \phi$. Thus an application of Theorem 2.3 shows that there is $\lambda > 0$ so that

$$\|P_{h,[-1,1]} - P_{g,[-1,1]}\|_{[-1,1]} \leqslant \lambda \|h - g\|_{[-1,1]} \tag{2.14}$$

for every $g \in C[-1, 1]$ and every $h \in \Gamma$. But (2.12) implies

$$\|P_{f_{k_j},[-1,1]} - g_{\hat{g}_{kj},[-1,1]}\|_{[-1,1]} > k_j \|f_{k_j} - \hat{g}_{k_j}\|_{[-1,1} \qquad j = 1, 2,...,$$

where $f_{k_j} \in \Gamma$ and $\hat{g}_{k_j} \in C[-1, 1]$ which contradicts (2.14). To prove (2.11) we apply Theorem 2.3 with $H = \pi_n$. Then there is a constant $\gamma > 0$ so that

$$\|h - P_{h,[-1,1]}\|_{[-1,1]} \leqslant \|h - p\|_{[-1,1]} - \gamma \|p - P_{h,[-1,1]}\|_{[-1,1]} \tag{2.15}$$

for every $p \in \pi_n$ and every $h \in \Gamma$. But (2.13) implies

$$\|f_{k_j} - P_{f_{k_j},[-1,1]}\|_{[-1,1]} > \|f_{k_j} - \hat{p}_{k_j}\|_{[-1,1]} - 1/k_j \|\hat{p}_{k_j} - P_{f_{k_j},[-1,1]}\|_{[-1,1]},$$

$j = 1, 2,...,$ where $f_{k_j} \in \Gamma$ and $\hat{p}_{k_j} \in C[-1, 1]$ which contradicts (2.15).

THEOREM 2.5. *Let $f \in C^{n+1}[-1, 1]$ so that $f^{(n+1)}(x) \neq 0$ for $x \in [-1, 0)$ or $x \in (0, 1]$. Suppose there are real numbers m, M, $0 < m \leqslant M$ and $p \in \pi_r$, $r \geqslant n$, so that*

$$0 \leqslant m\,|p^{(n+1)}(x)| \leqslant |f^{(n+1)}(x)| \leqslant M\,|p^{(n+1)}(x)|$$

on $[-\delta, \delta]$ for some $\delta > 0$. Then there are constants $\lambda_f > 0$ and $\gamma_f > 0$ so that for all closed intervals $J \subset [-1, 1]$

$$\|P_{f,J} - P_{g,J}\|_J \leqslant \lambda_f \|f - g\|_J \tag{2.16}$$

for all $g \in C(J)$ and

$$\|f - P_{f,J}\|_J \leqslant \|f - Q\|_J - \gamma_f \|Q - P_{f,J}\|_J \tag{2.17}$$

for all $Q \in \pi_n$.

Proof. If $f^{(n+1)}(0) \neq 0$ then Theorem 2.2 applies. Thus, suppose $f^{(n+1)}(0) = 0$. Since $f^{(n+1)}(x) \neq 0$ on $[-1, 0)$ and $(0, 1]$, $f \notin \pi_n$ on $[a, b]$ for any $[a, b] \subset [-1, 1]$. If such constants λ_f and γ_f do not exist then for every positive integer k, there is $J_k = [a_k, b_k] \subset [-1, 1]$ and $g_k \in C[a_k, b_k]$, $p_k \in \pi_n$ so that

$$\|P_{f,J_k} - P_{g_k,J_k}\|_{J_k} > k \|f - g_k\|_{J_k} \tag{2.18}$$

and

$$\|f - P_{f,J_k}\|_{J_k} > \|f - p_k\|_{J_k} - 1/k \, \|p_k - P_{f,J_k}\|_{J_k}. \tag{2.19}$$

We can choose a subsequence $\{J_{k_j}\}$ of the $\{J_k\}$ that converge to $[a, b]$, where $a \leqslant b$. If $a < b$, choose $\varepsilon = b - a$ and apply Lemma 2.4 to get contradictions to (2.18) and (2.19). Thus $a = b$. If $a = b \neq 0$ then for j sufficiently large, $0 \notin [a_{k_j}, b_{k_j}]$ and $f^{(n+1)}(x) \neq 0$ for $x \in [a_{k_j}, b_{k_j}]$. An application of Theorem 2.2 now gives the desired results. Thus, assume $a = b = 0$. Now define $q_{i,j}$ on $[a_{k_j}, b_{k_j}]$ for f as in Lemma 2.1 for $i = 1, 2,..., n + 2, j = 1, 2,...$. An application of steps (3.5) through (3.21) in the proof of Theorem 1 [4, pp. 229–231] gives (if $a_{k_j} < 0 < b_{k_j}$)

$$|q_{i,}(x)| \leqslant \frac{M\,|p^{(n+1)}(\xi)|\,(b_{k_j} - a_{k_j})^{n+1}}{m(n+1)!\,\max[E_n(p; a_{k_j}, 0), E_n(p; 0, b_{k_j})]} + 1 \tag{2.20}$$

or (if $a_{k_j} < b_{k_j} < 0$ or $0 < a_{k_j} < b_{k_j}$)

$$|q_{i,j}(x)| \leqslant \frac{M\,|p^{(n+1)}(\xi)|\,(b_{k_j} - a_{k_j})^{n+1}}{m(n+1)!\,E_n(p; a_{k_j}, b_{k_j})} + 1. \tag{2.21}$$

If (2.21) holds we can follow the procedure used in Theorem 2.2 to obtain (as in (2.7))

$$\|p^{(n+1)}\|_{[a_{k_j}, b_{k_j}]} \leqslant \frac{2^{n+1} r^{2n+2} E_n(p; a_{k_j}, b_{k_j})}{(b_{k_j} - a_{k_j})^{n+1}}. \tag{2.22}$$

Thus (2.21) and (2.22) imply

$$|q_{i,j}(x)| \leqslant \frac{M 2^{n+1} r^{2n+2}}{m(n+1)!} + 1. \tag{2.23}$$

If (2.20) holds and $\sup_{k_j}(b_{k_j} - a_{k_j})/|a_{k_j}| = \infty$ then by passing to a subsequence we can assume that $\lim_{j\to\infty}(b_{k_j} - a_{k_j})/|a_{k_j}| = \infty$. But then $(b_{k_j} - a_{k_j})/b_{k_j}$ is bounded and we can again follow the procedure of Theorem 2.2 to obtain

$$\|p^{(n+1)}\|_{[0, b_{k_j}]} \leqslant \frac{2^{n+1} r^{2n+2} E_n(p; 0, b_{k_j})}{b_{k_j}^{n+1}}. \tag{2.24}$$

Thus (2.20) and (2.24) together imply

$$|q_{i,j}(x)| \leqslant \frac{M 2^{n+1} r^{2n+2} (b_{k_j} - a_{k_j})^{n+1}}{m(n+1)!\, b_{k_j}^{n+1}} + 1. \tag{2.25}$$

Hence, (2.23) and (2.25) imply $\max_{1 \leqslant i < n+2} \|q_{i,j}\|_{J_{k_j}}$ is bounded for all i and

$$\lambda_{f,J_{k_j}} \leqslant 2 \max_{1 \leqslant i \leqslant n+2} \|q_{i,j}\|_{J_{k_j}}$$

and

$$\gamma_{f,J_{k_j}} \geqslant \left[\max_{1 \leqslant i \leqslant n+2} \|q_{i,j}\|_{J_{k_j}} \right]^{-1}.$$

But this contradicts (2.18) and (2.19) and our results follow.

3. Conclusions

The examples in [4] show that the hypotheses in the theorems of Section 2 cannot be weakened although Theorem 2.5 can be stated for a function having $n+1$ continuous derivatives whose $(n+1)$st derivative has a finite number of zeroes in $[-1, 1]$.

A potential application of theorems such as these is in the study of convergence of some of the adaptive curve fitting methods (e.g., see [6, 8]). With these techniques best approximations are computed on various subintervals by Remez type algorithms. The availability of a global strong unicity constant for all subintervals could be used to show convergence properties of the Remez algorithm independent of the subinterval on which it is applied.

References

1. W. M. Bartelt, On Lipschitz conditions, strong unicity, and a theorem of A. K. Cline, *J. Approx. Theory* **14** (1975), 245–250.
2. E. W. Cheney, "Introduction to Approximation Theory," McGraw–Hill, New York, 1966.
3. A. K. Cline, Lipschitz conditions on uniform approximation operators, *J. Approx. Theory* **8** (1973), 160–172.
4. M. S. Henry and J. A. Roulier, Uniform Lipschitz constants on small intervals, *J. Approx. Theory* **21** (1977), 224–235.
5. M. S. Henry and D. Schmidt, Continuity theorems for the product approximation operator, *in* "Theory of Approximation with Applications" (A. G. Law and B. N. Sahney, Eds.), pp. 24–42, Academic Press, New York, 1976.
6. J. A. Bull and G. D. Taylor, Adaptive curve fitting, manuscript.
7. G. G. Lorentz, "Approximation of Functions," Holt, Rinehart & Winston, New York, 1966.
8. G. D. Taylor, Data fitting: some adaptive methods, *in* "Approximation in Theory and Practice" (G. Meinardus, Ed.), pp. 291–304, Bibliographisches Institut, Mannheim, 1979.

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